PRELIMINARY INVESTIGATIONS OF A NONLINEAR DIFFERENTIAL EQUATION ARISING IN THE STUDY OF VORTEX GENERATION

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ABSTRACT

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Vortex shedding from a plate can be modeled by a nonlinear oscillator differential equation. We investigate the general properties of a certain approximation to this differential equation. In particular, detailed consideration is given to the number of possible limit-cycles and their stability properties. The method of slowly varying amplitude and phase is used for the mathematical analysis of the problem.

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CHAPTER ONE

INTRODUCTION

1.1. Statement of the Problem

The following nonlinear differential equation is considered

$$\frac{d^2x}{dt^2} + x = \epsilon\left[1 + \beta_1\left(\frac{dx}{dt}\right)^2 + \beta_2\left(\frac{dx}{dt}\right)^4\right]\frac{dx}{dt}, \tag{1.1}$$

where ϵ, β_1 and β_2 are parameters. The method of slowly varying amplitude and phase is used to analyze the possible solution behaviors for the condition

$$|\epsilon| \ll 1. \tag{1.2}$$

1.2. Summary of Thesis

Investigation of Eq. (1.1) shows that the differential equation can have a large number of possible solution behaviors. For example, both stable and unstable isolated periodic solutions (limit-cycles) can exist either together or alone. Solutions occur in which the number of periodic solutions is one, two or three. The state of rest, for all these cases, can be either stable or unstable.

1.3. Origin of the Differential Equation

The nonlinear differential equation

$$(1.3)\qquad \frac{d^2x}{dt^2} + A\,\frac{dx}{dt} + B\left(\frac{dx}{dt}\right)^3 + C\left(\frac{dx}{dt}\right)^5 + D\sin x = F\sin(\omega t + \phi),$$

where A, B, C, D and F are constants, and ω and ϕ are the (constant) frequency and phase angle, corresponds to a forced pendulum with an odd fifth-order damping force. This equation has been used to model vortex shedding from oscillating and rotating bodies in a parallel flow.[1-3] A simple example of such a physical system is the motion of a flat plate dropped in a fluid, such as air, with the flat surface of the plate held parallel to the ground.

The above differential equation has been investigated by the use of numerical integration techniques.[4] A variety of solution behaviors were observed. For the unforced motion, i.e., $F = 0$, both oscillation and rotation can occur. For $F \neq 0$, chaotic solutions were seen.

1.4. Dimensionless Equation

In Eq. (1.3), the various constants and the independent variable t have physical dimensions, i.e., they can be directly expressed in terms of units of time, length and mass. (Note that x has the units of angle, i.e., radians.) We now perform a series of transformation on Eq. (1.3) to put it in a form such that all constants that appear are "pure" numbers. This procedure is explained in detail in

Mickens.[5]

First, replace t by the dimensionless variable $\bar{t}$ where

$$(1.4) \qquad t \to \bar{t} = \sqrt{D}t, \qquad D > 0.$$

Consequently,

$$(1.5a) \qquad \left(\frac{A}{D}\right)\frac{dx}{dt} = \left[\frac{A\sqrt{D}}{D}\right]\frac{dx}{d\bar{t}} = A_1\frac{dx}{d\bar{t}},$$

$$(1.5b) \qquad \left(\frac{B}{D}\right)\left(\frac{dx}{dt}\right)^3 = B\sqrt{D}\left(\frac{dx}{d\bar{t}}\right)^3 = B_1\left(\frac{dx}{d\bar{t}}\right)^3,$$

$$(1.5c) \qquad \left(\frac{C}{D}\right)\left(\frac{dx}{dt}\right)^5 = CD^{3/2}\left(\frac{dx}{d\bar{t}}\right) = C_1\left(\frac{dx}{d\bar{t}}\right)^5,$$

where A_1, B_1 and C_1 are defined by comparison of the last two expressions on each line of Eqs. (1.5). This gives for Eq. (1.3)

$$(1.6) \qquad \ddot{x} + A_1\dot{x} + B_1(\dot{x})^3 + C_1(\dot{x})^5 + \sin x = F_1 \cos(\bar{\omega}\bar{t}+\phi),$$

where

$$(1.7) \qquad \bar{\omega} = \frac{\omega}{\sqrt{D}}, \qquad F_1 = \frac{F}{D}$$

and the following notation is used for the derivatives

$$(1.8)\qquad \dot{x} \equiv \frac{dx}{d\bar{t}}, \qquad \ddot{x} = \frac{d^2x}{d\bar{t}^2}.$$

With

$$(1.9)\qquad B_2 \equiv \frac{B_1}{A_1}, \qquad C_2 = \frac{C_1}{A_1},$$

Eq. (1.6) becomes

$$(1.10)\qquad \ddot{x} + \sin x + A_1\left[1 + B_2(\dot{x})^2 + C_2(\dot{x})^4\right]\dot{x} = F_1 \cos(\bar{\omega}\bar{t} + \phi).$$

This form of Eq. (1.3) is dimensionless, i.e., A_1, B_2, C_2, F_1, $\bar{\omega}$ and $\bar{t}$ are "pure" numbers. Another way of stating this result is to say that the new constants are ratios of other dimensional constants such that the physical units cancel for the ratios.

1.5. Outline of Thesis

Chapter Two presents the method of slowly varying amplitude and phase. Limit-cycles and limit-points are defined and discussed. The conditions for stability of periodic solution are given. The procedure is illustrated by means of two examples. Chapter Three is the major section of the thesis. It applies the method of slowly varying amplitude and phase to our model equation (1.1) after discussing how Eq. (1.1) is an approximation to the full Eq. (1.10). A detailed examination of the possible solution behaviors of Eq. (1.1) is given. Finally, in Chapter Four, additional future investigations are presented.

CHAPTER TWO

SLOWLY VARYING AMPLITUDE AND PHASE

2.1. Major Issues

Consider the nonlinear differential equation

$$\frac{d^2x}{dt^2} + x = \epsilon G\left(x, \frac{dx}{dt}\right), \tag{2.1}$$

where ϵ is a positive parameter and $G(x,\dot{x})$ is a nonlinear function of x and $\dot{x} \equiv dx/dt$. Except for very special forms of the function G, there do not exist general solutions to Eq. (2.1) that can be expressed in terms of a finite number of elementary functions.[5,6] In this Chapter a method will be described for calculating approximations to Eq. (2.1) under the following assumptions:

(a) ϵ is assumed to satisfy the condition

$$|\epsilon| \ll 1. \tag{2.2}$$

(b) $G(x,\dot{x})$ is assumed to be a polynomial function of x and $\dot{x}$.

The procedure to be presented is called the method of slowly varying amplitude and phase. It was invented nearly seventy years ago to deal with nonlinear differential equations of the form given by Eq. (2.1).[7] The rigorous mathematical basis for the method was provided by Krylov and his collaborators.[8]

The essential point is the realization that for $\epsilon = 0$, Eq. (2.1) has the exact solution

$$(2.3) \quad x(t) = A_0 \cos(t + \phi_0),$$

where A_0 and ϕ_0 are constants. For $\epsilon > 0$, the exact solution to Eq. (2.1) can always be expressed in the form

$$(2.4) \quad x(t,\epsilon) = A(\epsilon,t)\cos[t + \phi(\epsilon,t)],$$

where $A(\epsilon,t)$ and $\phi(\epsilon,t)$ are, for the moment, unknown functions of both ϵ and t. They satisfy the conditions

$$(2.5) \quad A(0,t) = A_0, \quad \phi(0,t) = \phi_0.$$

Since $x(t,\epsilon)$, the solution to Eq. (2.1) satisfies a second-order differential equation, the functions $A(\epsilon,t)$ and $\phi(\epsilon,t)$ must each satisfy first-order differential equations. In the next section, the equations for $A(\epsilon,t)$ and $\phi(\epsilon,t)$ are determined. From these equations, simpler relations are obtained for calculating approximate expressions for $A(\epsilon,t)$ and $\phi(\epsilon,t)$.

2.2. SVAP Procedure

The method of slowly varying amplitude and phase (SVAP) begins with the assumed solution form of Eq. (2.4) for Eq. (2.1). The essential point in determining equations for $A(\epsilon,t)$ and $\phi(\epsilon,t)$ is to impose the requirement that $x(t,\epsilon)$ satisfies the condition

$$(2.6) \qquad \frac{dx(t,\epsilon)}{dt} = -A(\epsilon,t)\sin[t + \phi(\epsilon,t)].$$

Note that for $\epsilon = 0$, this is just the result given by taking the derivative of Eq. (2.3). From this, first-order differential equations will be obtained for $A(\epsilon,t)$ and $\phi(\epsilon,t)$.

Differentiating Eq. (2.4) gives

$$(2.7) \qquad \frac{dx}{dt} = \frac{dA}{dt}\cos(t+\phi) - A\sin(t+\phi) - A\frac{d\phi}{dt}\sin(t+\phi).$$

For Eq. (2.7) to have the form of Eq. (2.6), we must require

$$(2.8) \qquad \frac{dA}{dt}\cos(t+\phi) - A\frac{d\phi}{dt}\sin(t+\phi) = 0.$$

If now the assumed derivative, Eq. (2.6) is differentiated, the following is obtained

$$(2.9) \quad \frac{d^2x}{dt^2} = -\frac{dA}{dt}\sin(t+\phi) - A\cos(t+\phi) - A\frac{d\phi}{dt}\cos(t+\phi).$$

Substituting the assumed solution, Eq. (2.4), its assumed derivative, Eq. (2.6), and the second derivative, given by Eq. (2.9) into the differential equation (2.1), we obtain

$$(2.10) \quad -\frac{dA}{dt}\sin\psi - A\cos\psi - A\frac{d\phi}{dt}\cos\psi$$
$$+ A\cos\psi = \epsilon G(A\cos\psi, -A\sin\psi),$$

or

$$(2.11) \quad \frac{dA}{dt}\sin\psi + A\frac{d\phi}{dt}\cos\psi = -\epsilon G(A\cos\psi, -A\sin\psi),$$

where

$$(2.12) \quad \psi(\epsilon,t) = t + \phi(\epsilon,t).$$

The two equations (2.8) and (2.11) can be solved for dA/dt and $d\phi/dt$ since they are linear in these expressions. If we do this, then the following expressions are obtained

$$(2.13) \quad \frac{dA}{dt} = -\epsilon G[A(\epsilon,t)\cos\psi(\epsilon,t), -A(\epsilon,t)\sin\psi(\epsilon,t)]\sin\psi,$$

$$(2.14) \quad \frac{d\phi}{dt} =$$
$$-\left[\frac{\epsilon}{A(\epsilon,t)}\right]G[A(\epsilon,t)\cos\psi(\epsilon,t), -A(\epsilon,t)\sin\psi(\epsilon,t)]\cos\psi$$

These are the desired first-order differential equations for $A(\epsilon,t)$ and $\phi(\epsilon,t)$. Note that these are the exact differential equations and are nonlinear and quite complicated in form. The first approximation of Krylov and Bogoliubov[8] will now be obtained.

Inspection of Eqs. (2.13) and (2.14) shows that the right-sides are periodic in the variable ψ with a period equal to 2π. Also, the derivatives are of order ϵ, i.e.,

$$(2.15)\quad \frac{dA}{dt} = O(\epsilon), \qquad \frac{d\phi}{dt} = O(\epsilon).$$

Thus, under the condition of Eq. (2.2), both A and ϕ are slowly varying functions of time. Consequently, they will change very little during the time $T = 2\pi$. If now, the right-sides of Eqs. (2.13) and (2.14) are averaged over the interval 2π, where the average of a function f(z) is taken to be

$$(2.16)\quad \left(\frac{1}{2\pi}\right)\int_0^{2\pi} f(z)dz \equiv \langle f\rangle_{av},$$

then the first approximation of Krylov and Bogoliubov is obtained for the function A and ϕ. These averaged expressions are

$$(2.17)\quad \frac{dA}{dt} = -\left(\frac{\epsilon}{2\pi}\right)\int_0^{2\pi} G(A\cos\psi, -A\sin\psi)\sin\psi\, d\psi$$

$$(2.18)\quad \frac{d\phi}{dt} = -\left[\frac{\epsilon}{2\pi A}\right]\int_0^{2\pi} G(A\cos\psi, -A\sin\psi)\cos\psi\, d\psi.$$

(In Eqs. (2.17) and (2.18), both A and ϕ should be indicated as being averaged functions. However, consistent with the literature on this topic,[5,6,8] the symbols are not shown.)

The differential equations (2.17) and (2.18) have the following structure

$$(2.19)\quad \frac{dA}{dt} = \epsilon H_1(A),$$

$$(2.20)\quad \frac{d\phi}{dt} = \epsilon H_2(A).$$

Note that the right-sides are functions only of A. Therefore, Eq. (2.19) can be solved for $A(\epsilon,t)$ since it is a separable differential equation. This value for $A(\epsilon,t)$ can then be substituted into Eq. (2.20) to give an expression that depends only on t. Consequently, it can be integrated directly. In Section 2.5, we illustrate the procedure with two simple examples.

2.3. Limit Cycles and Points

Special solutions of Eq. (2.19) are of particular importance. These are the stationary or constant solutions. Their values are determined by the roots to the equation

$$(2.21) \quad H_1(A) = 0.$$

Denote these constant (stationary) solutions by $\{\bar{A}_i\}$ where $i = 1,2,\ldots,N$; and N is the total number of roots to Eq. (2.21). For a given constant solution, call it $\bar{A}_\ell$ $(1 \le \ell \le N)$, Eq. (2.20) can be immediately integrated to give

$$(2.22) \quad \phi(\epsilon,t) = \phi_0 + \epsilon H_2(\bar{A}_\ell)t,$$

where ϕ_0 is an integration constant. This gives the corresponding approximation to Eq. (2.1) as [5,6,8]

$$(2.23) \quad x(t,\epsilon) = \bar{A}_\ell \cos\{[1 + \epsilon H_2(\bar{A}_\ell)]t + \phi_0\}.$$

This type of (approximate) periodic solution is called a <u>limit-cycle</u>. The amplitude of the oscillation is $\bar{A}_\ell$ and the frequency is

$$(2.24) \quad \omega(\epsilon,\bar{A}_\ell) = 1 + \epsilon H_2(\bar{A}_\ell).$$

Note that the frequency is in general a function of the amplitude.

If $A = 0$ is a solution of Eq. (2.21), then it is called a limit-point. It corresponds to a position of "rest" or an "equilibrium" state.

The above analysis indicates that a given system, as modeled by a differential equation of the form of Eq. (2.1), can have a number of limit-cycles as well as a limit-point. In the next section, we discuss the stability properties of the limit-cycles and limit-point.

2.4. Stability

Returning to Eq. (2.19), we see that if $H_1(A) > 0$, the amplitude increases, while for $H_1(A) < 0$, the amplitude decreases. From Eq. (2.19), we conclude that if the initial value of the amplitude A_0 is not stationary, i.e., $H_1(A_0) \neq 0$, then the amplitude will increase monotonically if $H_1(A_0) > 0$ and decrease monotonically if $H_1(A_0) < 0$. Therefore, with increase in time, the amplitude $A(\epsilon,t)$ will tend to a constant (stationary value). This means that any nonstationary oscillation approaches a stationary one with increase in time, i.e., starting from an arbitrary initial state, the system will generally approach either a limit-point or a limit-cycle.

We are now in a position to discuss the stability of the stationary periodic states. To proceed, we will examine a solution of Eqs. (2.19) and (2.20) that is close to $A(\epsilon,t) = \bar{A}_{\ell}$. Let

(2.25) $A(\epsilon,t) = \bar{A}_\ell + \eta(t),$

where

(2.26) $|\eta(0)| \ll \bar{A}_\ell.$

Substitution of Eq. (2.25) into Eq. (2.19) gives

(2.27) $\frac{d\eta}{dt} = \epsilon H_1(\bar{A}_\ell) + \epsilon \left.\frac{dH_1}{dA}\right|_{A=\bar{A}_\ell} \eta + \cdots$

Keeping only linear terms, using the fact that $H_1(\bar{A}_\ell) = 0$, and defining B as

(2.28) $B = \epsilon \left.\frac{dH_1}{dA}\right|_{A=\bar{A}_\ell},$

the equation for $\eta(t)$ becomes (in the linear approximation)

(2.29) $\frac{d\eta}{dt} = B\eta.$

This equation has the solution

(2.30) $\eta(t) = \eta_0 e^{Bt}.$

If $B < 0$, the stationary solution $\bar{A}_\ell$ is said to be stable; if $B > 0$, the stationary solution is unstable. Thus, the stability properties of a given stationary state depends on the sign of the derivative of $H_1(A)$ evaluated at the value of the stationary amplitude $\bar{A}_\ell$.

2.5. Two Elementary Applications

As illustrations of the method of slowly varying amplitude and phase, we will apply the procedures of the last section to two examples.

First, consider the equation

$$\text{(2.31)} \quad \frac{d^2x}{dt^2} + x + \epsilon x^3 = 0.$$

This corresponds to a conservative nonlinear oscillator in physics.[5] Comparison with Eq. (2.1) shows that

$$\text{(2.32)} \quad G = -x^3.$$

Therefore, substituting this expression into Eqs. (2.17) and (2.18) gives

$$\text{(2.33)} \quad \frac{dA}{dt} = -\left(\frac{\epsilon}{2\pi}\right)\int_0^{2\pi}(-)A^3(\cos\,\psi)^3\sin\,\psi\;d\psi = 0,$$

$$\text{(2.34)} \quad \frac{d\phi}{dt} = \left(\frac{\epsilon}{2\pi A}\right)\int_0^{2\pi} A^3\cos^4\psi\;d\psi.$$

From the first equation, we obtain

$$(2.35) \quad A(\epsilon,t) = A_0 = \text{constant.}$$

Consequently, Eq. (2.34) can be integrated to give

$$(2.36) \quad \phi(\epsilon,t) = \left(\frac{3\epsilon A_0^2}{8}\right)t + \phi_0.$$

Therefore, a first approximation to the solution of Eq. (2.31) is

$$(2.37) \quad x(t,\epsilon) = A_0 \cos\left[\left(1 + \frac{3\epsilon A_0^2}{8}\right)t + \phi_0\right].$$

For the second example consider the linear damped oscillator

$$(2.38) \quad \frac{d^2x}{dt^2} + 2\epsilon \frac{dx}{dt} + x = 0.$$

For this case

$$(2.39) \quad G = -2 \frac{dx}{dt}.$$

An easy, direct calculation gives

$$(2.40) \quad \frac{dA}{dt} = -\epsilon A$$

$$(2.41) \quad \frac{d\phi}{dt} = 0.$$

Solutions of these equations are trivial to obtain. Substitution into Eq. (2.4) gives

$$(2.42)\quad x(t,\epsilon) = A_0 e^{-\epsilon t} \cos(t+\phi_0),$$

where A_0 and ϕ_0 are arbitrary constants. This can be compared to the exact solution of Eq. (2.38) which is

$$(2.43)\quad x(t,\epsilon) = A_0 e^{-\epsilon t} \cos\left[\left(1 - \frac{\epsilon^2}{4}\right)^{1/2} t + \phi_0\right].$$

CHAPTER THREE

SMALL AMPLITUDE OSCILLATIONS

3.1. The Differential Equation

The forced nonlinear differential equation (1.10) is intractable for almost any kind of mathematical procedure for calculating approximate analytical solutions.[5,6,8,9] For example, it clearly does not fit into the format of Eq. (2.1) for which the method of slowly varying amplitude and phase (SVAP) holds.

To proceed, we consider the unforced case of Eq. (1.10), i.e., $F_1 = 0$. Furthermore, the small amplitude approximation will be imposed on the sin x term, i.e., sin x will be replaced by its linear approximation

$$(3.1) \quad \sin x \simeq x.$$

Under these conditions Eq. (1.10) becomes

$$(3.2) \quad \ddot{x} + x = \epsilon[1 + \beta_1(\dot{x})^2 + \beta_2(\dot{x})^4]\dot{x},$$

where

$$(3.3) \quad \epsilon = -A_1, \quad \beta_1 = B_1, \quad \beta_2 = C_1.$$

With the assumption

$$(3.4) \quad |\epsilon| \ll 1,$$

then Eq. (3.2) is of a form such that the SVAP method can be used to determine approximations to its solution.

It is important to state that practical application of the SVAP method shows that while it was obtained under the conditions of small values for ϵ, it provides an excellent description of the possible solution behaviors also for ϵ large.[6,8,10] A second point of equal importance is the fact that the SVAP method is not limited to small values of $x(t)$. It can be applied to situations where $x(t)$ can become "large." The critical issue, as presented in Section 2.2, is the requirement that the derivatives of the amplitude and phase be small; see Eqs. (2.19) and (2.20). Thus, $x(t)$ and $dx(t)/dt$ may be large and the procedure still provides a good approximation to the exact solution.[10]

To illustrate the SVAP method, we now consider a system that has one limit-cycle and one limit-point. The system equation is

$$(3.5) \quad \ddot{x} + x = \epsilon(1-x^2)\dot{x},$$

and is called the van der Pol equation.[7] (It was invented in 1922 to model a nonlinear oscillating electrical circuit.) This will provide not only a better understanding of the SVAP method, but, also lead directly into the analysis of the differential equation of this thesis, namely, Eq. (3.2). Comparison of Eq. (3.2) with Eq. (3.5) shows that Eq. (3.2) may be considered a generalization of the van der Pol equation.

For the van der Pol differential equation the nonlinear function $G(x,\dot{x})$ is

$$(3.6) \quad G(x,\dot{x}) = (1-x^2)\dot{x}.$$

The corresponding amplitude and phase equations are

$$(3.7) \quad \frac{dA}{dt} = \left(\frac{\epsilon}{2\pi}\right)\int_0^{2\pi}(1 - A^2 \cos^2\psi)A \sin^2\psi \, d\psi,$$

$$(3.8) \quad \frac{d\phi}{dt} = \left(\frac{\epsilon}{2\pi A}\right)\int_0^{2\pi}(1 - A^2 \cos^2\psi)A \sin\psi \cos\psi \, d\psi.$$

These relations follow, respectively, from Eqs. (2.17) and (2.18). Using the trigonometric relations

$$(3.9a) \quad \cos^2\psi = (1 + \cos 2\psi)/2, \qquad \sin^2\psi = (1 - \cos 2\psi)/2,$$

$$(3.9b) \quad \sin\psi_1 \cos\psi_2 = [\sin(\psi_1+\psi_2) + \sin(\psi_1-\psi_2)]/2,$$

$$(3.9c) \quad \cos\psi_1 \cos\psi_2 = [\cos(\psi_1+\psi_2) + \cos(\psi_1-\psi_2)]/2,$$

$$(3.9d) \quad \sin\psi_1 \sin\psi_2 = [\cos(\psi_1-\psi_2) - \cos(\psi_1+\psi_2)]/2,$$

$$(3.9e) \quad \int_0^{2\pi} \cos(n\psi)d\psi = \int_0^{2\pi} \sin(n\psi)d\psi = 0, \text{ for } n = 1,2,3,\ldots,$$

Eqs. (3.7) and (3.8) become

$$(3.10)\quad \frac{dA}{dt} = \left(\frac{\epsilon A}{2}\right)\left(1 - \frac{A^2}{4}\right)$$

$$(3.11)\quad \frac{d\phi}{dt} = 0.$$

Let $A(0) = A_0$ and $\phi(0) = \phi_0$. Then the solution to Eq. (3.11) is

$$(3.12)\quad \phi(\epsilon,t) = \phi_0.$$

Multiplying both sides of Eq. (3.10) by 2A and rewriting in terms of the new variable $z = A^2$ gives

$$(3.13)\quad 2A\,\frac{dA}{dt} = \epsilon A^2\left(1 - \frac{A^2}{4}\right)$$

and

$$(3.14)\quad \frac{dz}{dt} = \epsilon z\left(1 - \frac{z}{4}\right).$$

This equation may be directly integrated to give

$$(3.15)\quad z(t) = [A(t)]^2 = \frac{4A_0^2}{A_0^2 + (4 - A_0^2)e^{-\epsilon t}}.$$

Therefore, according to the SVAP method, a first approximation to the solution of the van der Pol equation is

$$(3.16)\quad x(t) = A(\epsilon,t)\cos[t + \phi(\epsilon,t)]$$

where on using the results from Eqs. (3.12) and (3.15), the following expression is obtained

$$(3.17)\quad x(t) = \frac{2A_0 \cos(t + \phi_0)}{\sqrt{A_0^2 + (4-A_0^2)e^{-\epsilon t}}} .$$

Note that as $t \to \infty$, $A(t) \to 2$, and the solution becomes

$$(3.18)\quad x(t) = 2 \cos(t + \phi_0).$$

Returning to Eq. (3.10) and comparing it to Eq. (2.19) gives

$$(3.19a)\ H_1(A) = \left(\frac{\epsilon A}{2}\right)\left(1 - \frac{A^2}{4}\right).$$

The limit-cycles and limit-points are solutions to $H_1(A) = 0$. They are

$$(3.19b)\ \bar{A}_1 = 0, \quad \bar{A}_2 = 2.$$

$\bar{A}_1 = 0$ corresponds to a limit-point, i.e., the solution (to this approximation) is $x(t) = 0$, which is the equilibrium or resting state of the system. However, $\bar{A}_2 = 2$ corresponds to a limit-cycle; see Eq. (3.18). The corresponding periodic solution has amplitude two and a frequency of $\omega = 1$.

The stability properties are determined by calculating the derivative of $H_1(A)$, i.e.,

$$(3.20a)\quad \frac{dH_1(A)}{dA} = \left(\frac{\epsilon}{2}\right)\left(1 - \frac{3A^2}{4}\right)$$

and evaluating it at $A = 0$ and $A = 2$. Doing this gives

$$(3.20b)\quad \frac{dH(0)}{dA} = \frac{\epsilon}{2},$$
$$(3.20c)\quad \frac{dH(2)}{dA} = -\epsilon.$$

Using the analysis of Section 2.4, the following conclusions are reached:

(a) If $\epsilon > 0$, the limit-point is unstable and the limit-cycle is stable. This means that if $x_0 = 0$ and $\dot{x}_0 = 0$, then the system remains in the rest state for all time. If $x(t)$ is perturbed from the rest state or if $0 < x_0 < 2$, then the amplitude increases monotonically to the value two. The motion is then given by Eq. (3.18). Likewise, if $\dot{x}_0 = 0$ and $x_0 > 2$, the motion is oscillatory with an amplitude that decreases to the value two. Again, the limit-cycle behavior of Eq. (3.18) is obtained as $t \to \infty$. Thus, for this case, all motion approaches the limit-cycle.

(b) If $\epsilon < 0$, the limit-point is stable and the limit-cycle is unstable. For this case, any initial state with

$0 < x_0 < 2$ and $\dot{x}_0 = 0$, eventually approaches the equilibrium state $x(t) = 0$. Any motion with $x_0 > 2$ and $\dot{x}_0 = 0$ "blows-up," i.e., the motion becomes unbounded.

Examination of Eq. (3.17) shows the behavior of the solutions as discussed in (a) and (b).

3.2. Approximate Analytic Solution

The differential equation (3.2) is of the form given by Eq. (2.1) with

$$(3.21)\quad G(x,\dot{x}) = [1 + \beta_1(\dot{x})^2 + \beta_2(\dot{x})^4]\dot{x}.$$

Thus, the SVAP method can be applied to determine an approximation to the exact solution.

From Eq. (2.18), the equation for the phase $\phi(\epsilon,t)$ is

$$(3.22)\quad \frac{d\phi}{dt} = -\left(\frac{\epsilon}{2\pi A}\right)\int_0^{2\pi} G(A\cos\psi,\ -A\sin\psi)\cos\psi\ d\psi,$$

where

$$(3.23)\quad \left(\frac{-\epsilon}{2\pi A}\right)G(A\cos\psi,\ -A\sin\psi)\cos\psi = \left(\frac{\epsilon}{2\pi}\right)[1 + \beta_1 A^2\sin^2\psi + \beta_2 A^4\sin^4\psi]\cos\psi.$$

Using the trigonometric relations

$$(3.24a)\quad \sin^3\psi = \left(\tfrac{3}{4}\right)\sin\psi - \left(\tfrac{1}{4}\right)\sin 3\psi,$$

$$(3.24b)\quad \sin^5\psi = \left(\tfrac{10}{16}\right)\sin\psi - \left(\tfrac{5}{16}\right)\sin 3\psi + \left(\tfrac{1}{16}\right)\sin 5\psi,$$

and the relations given by Eqs. (3.9), it is easily seen that upon integration of the right-side of Eq. (3.22), the following result is obtained

$$(3.25)\quad \frac{d\phi}{dt} = 0.$$

Therefore

$$(3.26)\quad \phi(\epsilon,t) = \phi_0,$$

and the phase is constant.

Likewise the equation for the amplitude $A(\epsilon,t)$ is, from Eq. (2.17), given by the expression

$$(3.27)\quad \frac{dA}{dt} = -\left(\frac{\epsilon}{2\pi}\right)\int_0^{2\pi} G(A\cos\psi,\ -A\sin\psi)\sin\psi\, d\psi,$$

where

$$(3.28)\quad \left(\frac{-\epsilon}{2\pi}\right)G(A\cos\psi,\ -A\sin\psi)\sin\psi$$
$$= \left(\frac{\epsilon A}{2\pi}\right)[1 + \beta_1 A^2\sin^2\psi + \beta_2 A^4\sin^4\psi]\sin^2\psi.$$

Using the following trigonometric relations

(3.29a) $\sin^4\psi = (\frac{3}{8}) - (\frac{1}{2})\cos 2\psi + (\frac{1}{8})\cos 4\psi,$

(3.29b) $\sin^6\psi = (\frac{5}{16})-(\frac{15}{32})\cos 2\psi + (\frac{3}{16})\cos 4\psi - (\frac{1}{32})\cos 6\psi,$

and the relations given by Eqs. (3.9), it is easily seen that

$$(3.30)\quad \left(\frac{-\epsilon}{2\pi}\right)G \sin \psi = \left(\frac{\epsilon A}{4\pi}\right)\left[1 + \left(\frac{3\beta_1}{4}\right)A^2 + \left(\frac{5\beta_2}{8}\right)A^4\right]$$

$+$ [trigonometric terms in $\sin 2\psi$, $\sin 4\psi$ and $\sin 6\psi$].

Substitution of Eq. (3.30) into the right-side of Eq. (3.27) and integrating gives

$$(3.31)\quad \frac{dA}{dt} = \left(\frac{\epsilon A}{2}\right)\left[1 + \left(\frac{3\beta_1}{4}\right)A^2 + \left(\frac{5\beta_2}{8}\right)A^4\right].$$

Introducing the new variable

$$(3.32)\quad z = A^2,$$

Eq. (3.31) can be rewritten to the form

$$(3.33)\quad \frac{dz}{dt} = \epsilon z\left[1 + \left(\frac{3\beta^1}{4}\right)z + \left(\frac{5\beta_2}{8}\right)z^2\right].$$

The analysis presented in Sections 2.3 and 2.4 can now be applied to Eq. (3.33). This will allow a determination of the stability properties of the limit-point at $z = 0$, and of the number and stability properties of any limit-cycles.

The use of the variable z is of particular value since the right-side of the determining equation for z is essentially quadratic, while that for A, see Eq. (3.31), is quartic. (Of course, the actual full order of the polynomial expressions on the right-sides of Eq. (3.33) and (3.31) are, respectively, of cubic and fifth order. However, one root, $z = 0$, is known.) Since $z = A^2$ and A must be real to correspond to a limit-cycle, the only roots of the expression

$$(3.34) \quad \bar{H}(z) = \epsilon z\left[1 + \left(\frac{3\beta_1}{4}\right)z + \left(\frac{5\beta_2}{8}\right)z^2\right] = 0,$$

that indicate limit-cycle behavior are those for which z is both real and positive. Other roots, i.e., z either negative or complex, give complex values for the amplitude A. Consequently, as stated above, they cannot correspond to limit-cycles.

The next two sections will contain the results of this analysis of Eq. (3.33). The analysis will not depend on a knowledge of the actual values of the constants ϵ, β_1 and β_2. The goal is to obtain information on the possible solution behaviors of the differential equation (3.2).

3.3. Origin Unstable ($\epsilon > 0$)

Equation (3.33) can be written as

$$(3.35) \quad \frac{dz}{dt} = \bar{H}(z),$$

using the result given by Eq. (3.34). The following points are of particular importance:

(a) When $\bar{H}(z) > 0$, $z(t)$ is an increasing function of t.

(b) When $\bar{H}(z) < 0$, $z(t)$ is a decreasing function of t.

(c) For $\bar{H}(z) = 0$, $z(t)$ is stationary, i.e., $z(t)$ is a constant. These values of z correspond to the limit-cycle motions of Eq. (3.2).

(d) The slope of $\bar{H}(z)$ at $z = 0$ is ϵ.

(e) $z = \bar{z}_1 = 0$ is a zero of the equation $\bar{H}(z) = 0$ for all values of ϵ, β_1 and β_2. This is a limit-point of the motion. Consequently, Eq. (3.2) always has $x(t) = 0$ as a resting or equilibrium state. Furthermore, from (d), if $\epsilon > 0$, the limit-point is unstable; if $\epsilon < 0$, the limit-point is stable.

In this section, the situation for $\epsilon > 0$ will be considered. For this case, the limit-point $\bar{z}_1 = 0$ is unstable. The condition

$$(3.36) \quad \beta_2 \neq 0,$$

will be assumed to hold, otherwise Eqs. (3.2) and (3.33) reduce to the relations of the van der Pol equation.

The procedure to be used to analyze Eq. (3.33) will be geometric rather than algebraic; that is, Eq. (3.33) will not be solved directly for its three roots in terms of β_1 and β_2. This could easily be done since one root, $\bar{z}_1 = 0$, is known and the remaining expression is a quadratic equation. However, very little information would be obtained in this manner without explicit knowledge of the values of β_1 and β_2. The geometrical approach permits a direct pictorial representation of how the amplitude $A = \sqrt{z}$ changes as a function of t. (Again, it should be stated that the goal of this thesis is to determine the various solution behaviors for Eq. (3.2).)

To begin, under the conditions $\epsilon > 0$ and $\beta_2 \neq 0$, the cubic equation $\bar{H}(z)$ has seven possible distinct "shapes." Each different "shape" will lead to a unique type of solution behavior for $z(t)$ and, thus, for $A(\epsilon,t)$. From Eq. (3.26), the phase $\phi(\epsilon,t)$ is constant; therefore, the (approximate) solution to Eq. (3.2) takes the form

$$(3.37) \quad x(t) = A(\epsilon,t)\cos(t + \phi_0).$$

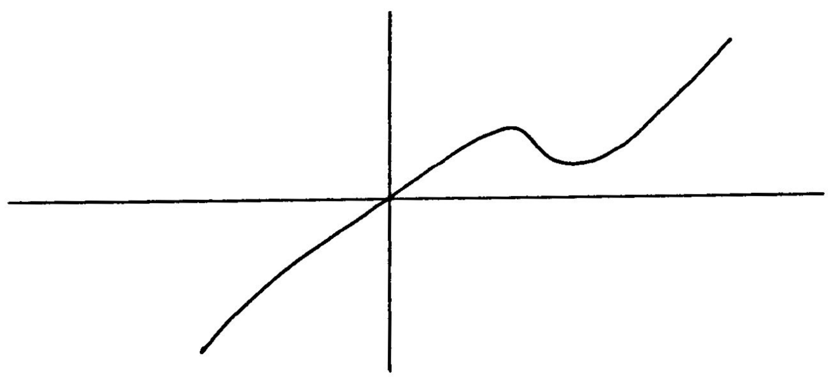
This equation clearly shows that an analysis of $z(t)$ gives the behavior of $x(t)$.

In the "pictorial equations" presented below, $\bar{H}(z)$ is plotted vs z. However, in general, no explicit indication of the labelling of the vertical and horizontal axes will be shown.

Case 1

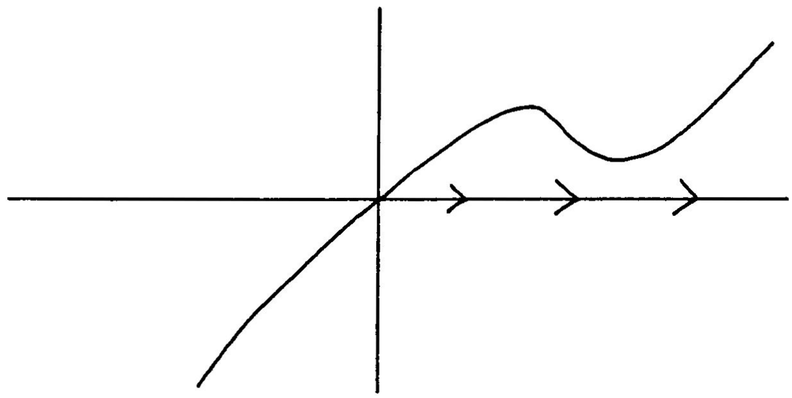
The cubic curve takes the form

(3.38)



This corresponds to $\bar{H}(z)$ having one real root at $z = \bar{z}_1 = 0$ and two complex (conjugate) roots. Since, for $z > 0$, $\bar{H}(z) > 0$, the derivative dz/dt is always positive; see Eq. (3.35). Consequently, $A(\epsilon,t)$ is an increasing function of t. The solution, Eq. (3.37), for this case is oscillatory with an increasing amplitude. The motion is unstable for any initial amplitude, i.e., with increase of t, the amplitude increases monotonically. Hence, for this case, all motions are unstable.

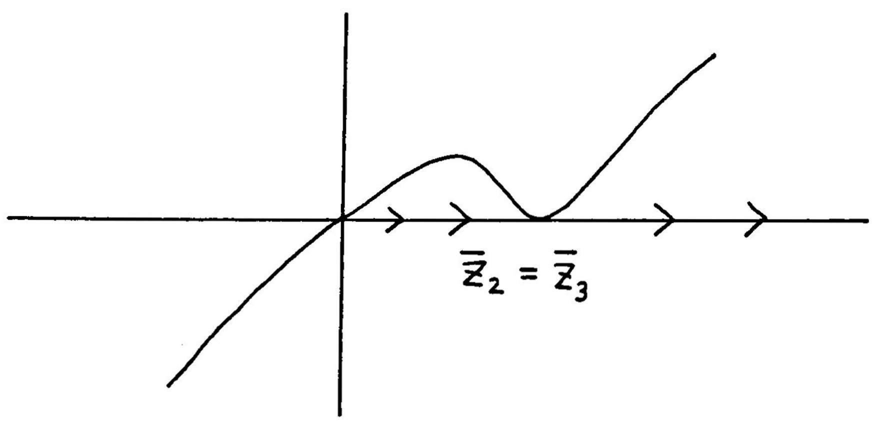
Additional information can be gotten from Eq. (3.38) if arrows are drawn indicating the directions of increasing or decreasing z. For this case, the following is obtained

(3.39) 

The arrows indicate that for $z > 0$, all motions correspond to increasing $z(t)$.

Case 2

The cubic curve has the form

(3.40) 

For this case, all the roots are real and non-negative, with $\bar{z}_1 = 0$ and two equal roots, $\bar{z}_2 = \bar{z}_3$. The non-zero roots give rise to a semi-stable limit-cycle. For $0 < z <$

$\bar{z}_2$, the amplitude increases and for $z > \bar{z}_2$, the amplitude also increases. It is stationary for $z = \bar{z}_2$.

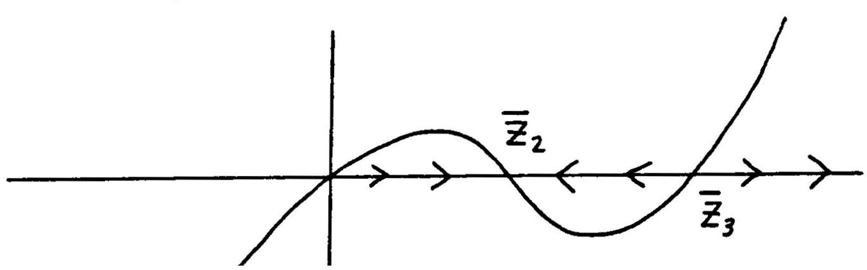
If $z_0 = z(0)$ is selected such that $0 < z_0 < \bar{z}_2$, the amplitude will increase until it reaches (asymptotically) the value $z = \bar{z}_2$. It will remain in this state indefinitely. However, if perturbed, such that the new value of z is greater than $\bar{z}_2$, the amplitude will increase (without limit) monotonically. If perturbed, such that the new value of z is less than $\bar{z}_2$, the amplitude will increase monotonically to $\bar{z}_2$; hence, the name semi-stable limit-cycle.

Cases 1 and 2 have been discussed in detail to show the nature of our analysis procedure. The additional cases to follow, in this and the next section, will not present such detail. However, it should be clear what is being done.

<u>Case 3</u>

The cubic equation takes the form

(3.41)

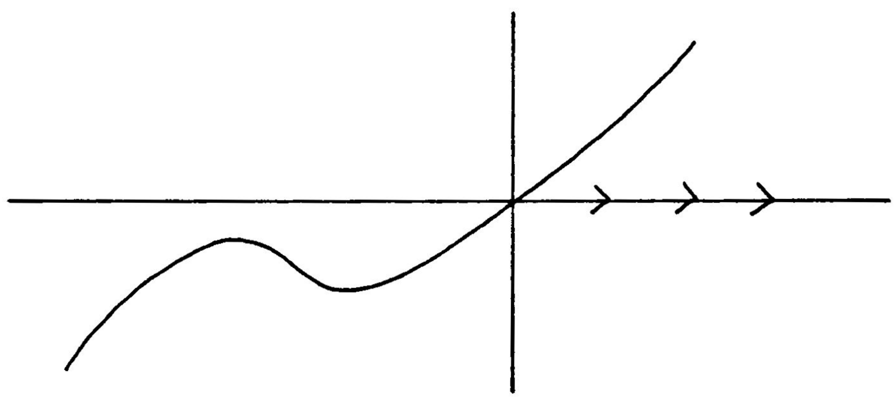


For this situation, $\bar{z}_2$ corresponds to a stable limit-cycle, while $\bar{z}_3$ is an unstable limit-cycle. Any initial condition in the interval, $0 < z_0 < \bar{z}_3$, leads (asymptotically) to the stable limit-cycle at $\bar{A}_2 = \sqrt{\bar{z}_2}$. For $z_0 > \bar{z}_3$, the amplitude increases monotonically with t.

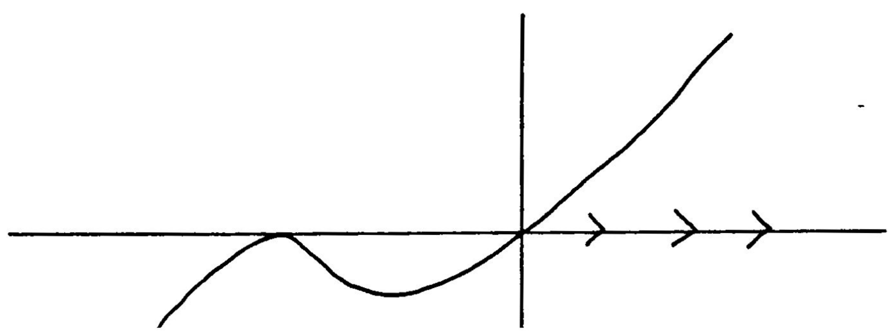
<u>Cases 4, 5. 6</u>

The cubic equations for these three cases are

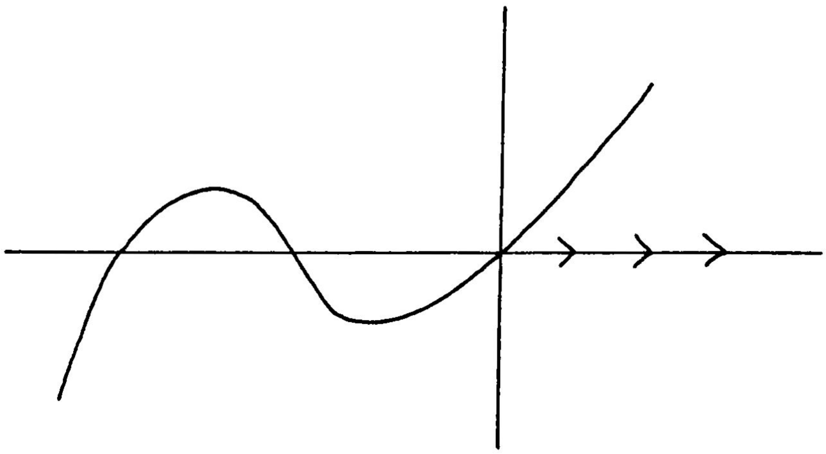
(3.42a)



(3.42b)



(3.42c)

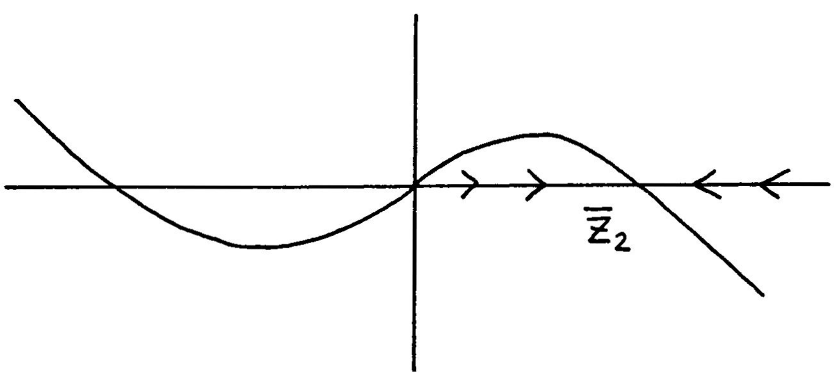


For all three of these cases, the roots of $\bar{H}(z) = 0$ correspond to complex values of A except for the real root at $\bar{z}_1 = 0$. Thus, the limit-point x(t) is unstable and all motions correspond to the amplitude $A(\epsilon,t)$ increasing monotonically, i.e., the total motion is unstable.

Case 7

The cubic equation takes the form

(3.43)



For this case, there is a stable limit-cycle at $\bar{A}_2 = \sqrt{\bar{z}_2}$. The origin is unstable. For $z_0 > 0$, all motions (asymptotically) approach the limit-cycle given by

$$(3.44) \quad x(t) = \sqrt{\bar{z}_2} \cos(t+\phi_0).$$

The system, for this case, has global stability.

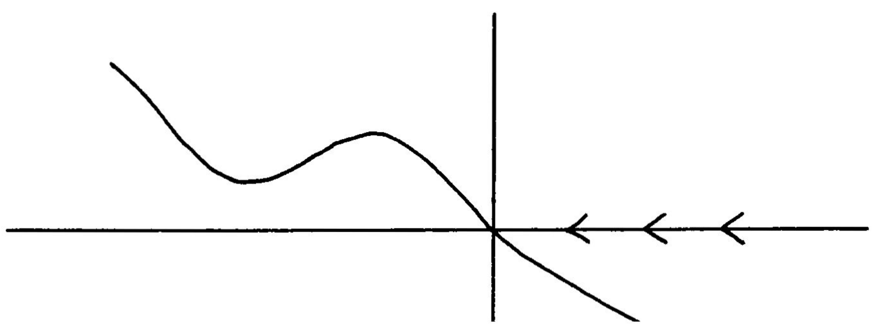
3.4. Origin Stable ($\epsilon < 0$)

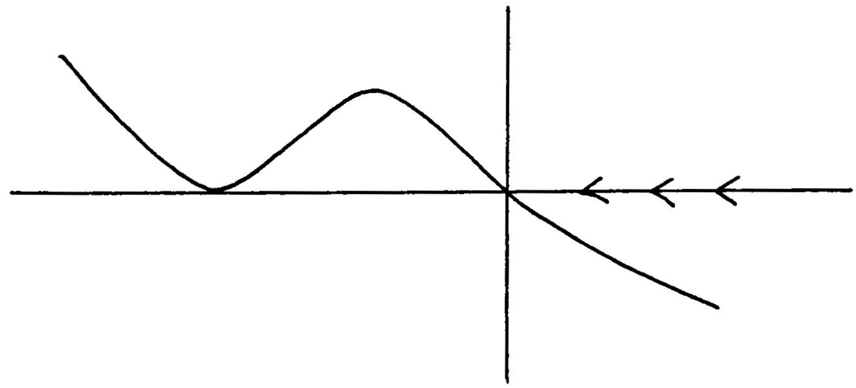
In Section 3.3, the limit-point or origin was unstable. This section investigates the various cases corresponding to the origin being stable i.e., $\epsilon < 0$. The notations and conventions of Section 3.3 are used in this section. Again, there are seven cases to consider.

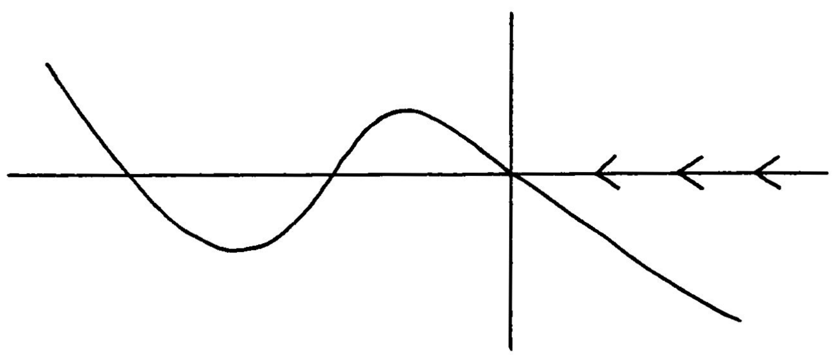
Cases 8, 9, 10

The cubic equation has one of the forms

(3.45a)



(3.45b) 

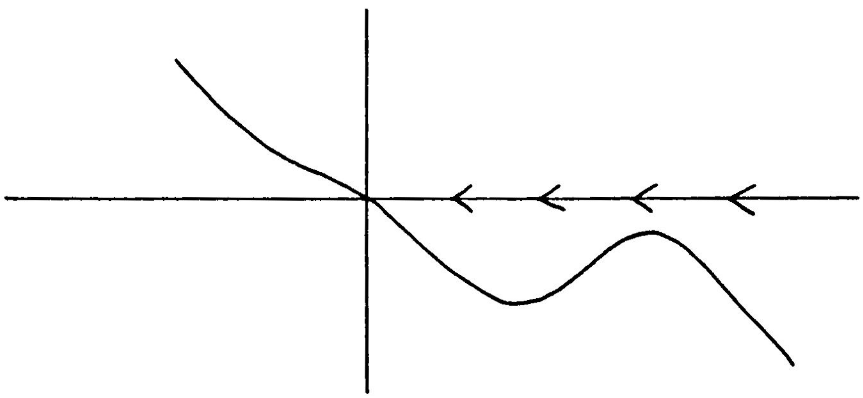
(3.45c) 

For each of these three situations the roots of $\bar{H}(z) = 0$, except for $\bar{z}_1 = 0$, are either real and negative or complex. Thus, the only real value of A is $\bar{A}_1 = 0$ which corresponds to a stable limit-point. All the motions are globally stable, i.e., if $z_0 > 0$, then $z(t)$ and, consequently, $A(\epsilon,t)$, decreases monotonically to zero with increase of t.

Case 11

The cubic equation takes the form

(3.46)

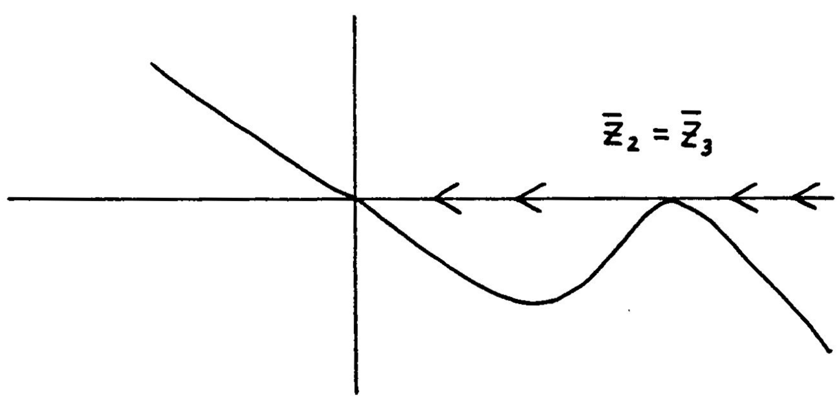


For this case, $\bar{H}(z)$ has two complex roots and the real root $\bar{z}_1 = 0$. The motion is global.

Case 12.

The cubic equation takes the form

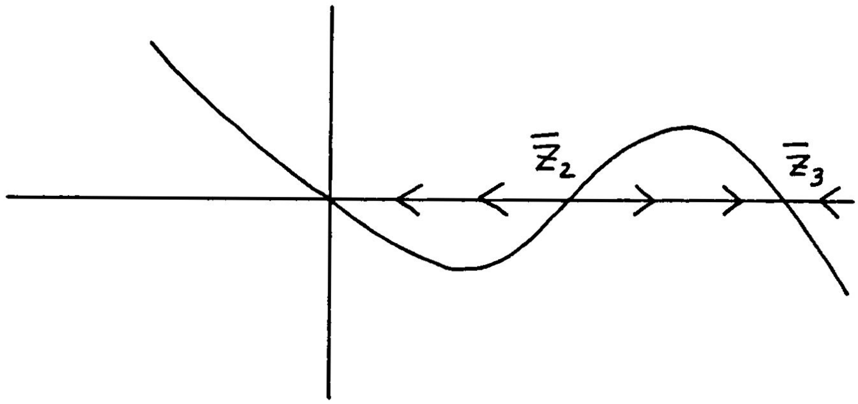
(3.47)



All the roots to $\bar{H}(z) = 0$ are non-negative with two roots equal, i.e., $\bar{z}_2 = \bar{z}_3$. As was the situation for Case 2 of the last section, the limit-cycle is semi-stable. The total motion is globally stable.

Case 13

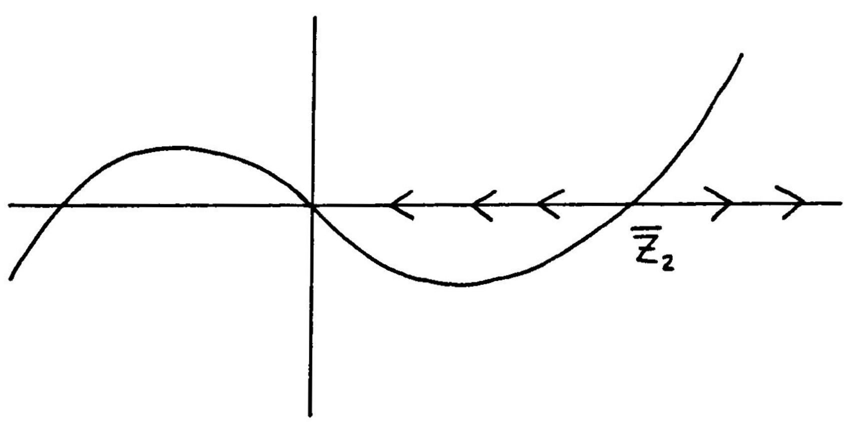
For this case the cubic equation takes the form

(3.48) 

All the roots of $\bar{H}(z) = 0$ are non-negative. The origin, $\bar{z}_1 = 0$, is stable; there are two limit-cycles: one at $\bar{z}_2$ which is unstable and one at $\bar{z}_3$ which is stable. The system has global stability.

Case 14

The cubic equation has the form

(3.49) 

All the roots of $\bar{H}(z) = 0$ are real: $\bar{z}_1 = 0$, $\bar{z}_2 > 0$ and $\bar{z}_3 < 0$. The origin is stable and the limit-cycle corresponding to $\bar{z}_2$ is unstable.

In the first section of the next chapter, a summary of the results derived in Sections 3.3 and 3.4 will be given and discussed.

CHAPTER FOUR

SUMMARY AND FUTURE INVESTIGATIONS

4.1 Summary and Discussion

Chapter Three, in particular the last two sections, contains a number of results critical to the understanding of the possible solution behaviors of Eq. (3.2). The following is a concise summary of what was obtained.

$\epsilon > 0$

For this situation, the origin or limit-point, $\bar{z}_1 = A_1^2 = 0$, is unstable. The solution possibilities for this case include the behaviors:

(a) there are no limit-cycles and all motions are unstable;

(b) there is one semi-stable limit-cycle and the system is globally unstable;

(c) near the origin is a stable limit-cycle; further out is an unstable limit-cycle; the system is globally unstable;

(d) there is a single stable limit-cycle; the global motion is stable.

$\epsilon < 0$

The origin or limit-point $\bar{z}_1 = A_1^2 = 0$ is stable. The solution possibilities for this case include the behaviors:

(a) there are no limit-cycles and all motions are stable;

(b) there exists one semi-stable limit-cycle and the system is globally stable;

(c) there exists an unstable limit-cycle and the system is globally unstable;

(d) there is one unstable limit-cycle near the origin and one stable limit-cycle further out from the origin; the system is globally stable.

Since the number of real, non-negative roots of $\bar{H}(z) = 0$ depends on the parameters β_1 and β_2, and the stability of any limit-cycles and the limit-point at $\bar{z}_1 = 0$ depends also on ϵ, it is clear that by proper choice of these three parameters the system described by Eq. (3.2) can be made to have any number of time dependent behaviors. In particular, the parameters of the system can be selected such that all motions decay with time; this corresponds to the case (a) of the $\epsilon < 0$ situation. Often this possibility is the one that is desirable in many scientific and technological applications.[11]

4.2. Related Research Problems

The nonlinear differential equation considered in this thesis, Eq. (3.2), is but an approximation to the more complicated forced Eq. (1.10). This differential equation has relevance as a model to simulate phenomena for which vortex shedding from oscillating and rotating bodies occurs.[1-3] This forced differential equation has a complex variety of new possible solution behaviors. These include both harmonic, subharmonic, superharmonic and chaotic solutions.[12] Preliminary results have already been obtained.[4] However, these studies need to be extended to cover a greater range of parameter values. Quantities to be determined include the fractal dimensions of strange attractors[12] and the spectral properties of the various types of solutions. For these studies numerical procedures are needed since no general mathematical theory exists that can be applied to the analysis of Eq. (1.10).

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